Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
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doi:10.55730/1300-0098.

Ranks of certain semigroups of transformations with idempotent complement whose restrictions belong to a given semigroup

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Received: 11.04.2023 • **Accepted/Published Online:** 29.02.2024 • **Final Version:** ..202

Abstract: For $n \geq 2$, let P_n, I_n, T_n, and S_n be the partial transformation semigroup, symmetric inverse semigroup, (full) transformation semigroup, and symmetric group on the set $X_n = \{1, \ldots, n\}$, respectively. In this paper, we find the ranks of certain subsemigroups of P_n, I_n, and T_n consisting of transformations with idempotent complement whose restrictions to the set X_m belong to the (possible) semigroup S_m, I_m, T_m, or P_m for $1 \leq m \leq n-1$.

Key words: Partial (full) transformation semigroup, symmetric inverse semigroup, symmetric group, idempotent, rank

1. Introduction

For $n \geq 2$, let P_n, I_n, T_n, and S_n be the partial transformation semigroup, symmetric inverse semigroup, (full) transformation semigroup, and symmetric group, on the set $X_n = \{1, \ldots, n\}$, respectively.

As it is well-known from Cayley's theorem for finite groups that every finite group is isomorphic to a subgroup of a symmetric group S_n. Similarly, it is well-known that every finite semigroup is isomorphic to a subsemigroup of a finite transformations semigroup T_n, and that every finite inverse semigroup is isomorphic to a subsemigroup of a finite symmetric inverse semigroup I_n. Another well-known fact is the semigroup P_n and the subsemigroup P_n^* of the transformations semigroup consisting of all self maps on $X_n \cup \{0\}$ for which $0\alpha = 0$ are isomorphic. Hence, the importance of T_n and P_n to finite semigroup theory, and the importance of I_n to finite inverse semigroup theory, may be likened to the importance of symmetric group S_n to finite group theory. Therefore, these semigroups are important research topics for researchers and there are many studies on these semigroups and their subsemigroups (see, for example, [3, 6, 11, 13]).

Let S be a semigroup and let $\emptyset \neq A \subseteq S$. Then, the smallest subsemigroup of S containing A, the semigroup consisting of all finite products of elements from A, is called the subsemigroup generated by A and denoted by $\langle A \rangle$. If S is finitely generated, that is there exists a finite $\emptyset \neq A \subseteq S$ such that $S = \langle A \rangle$, then the positive integer

$$\operatorname{rank}(S) = \min\{\, |A| : \langle A \rangle = S \text{ and } |A| < \infty \,\}$$

is called the rank of S. Moreover, any generating set of S with cardinality $\operatorname{rank}(S)$ is called the minimal generating set of S. The problem of finding any minimal generating set and so the rank of a semigroup, similar to the problem of finding the dimension of a group in group theory, is an interesting and important problem for

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2010 *AMS Mathematics Subject Classification:* 20M20

researchers on semigroups. Therefore, there are a lot of studies on various generating sets and ranks of certain semigroups (see, for example, [2, 4, 8, 17]).

For $\alpha \in P_n$, the *domain*, *image*, *fix*, and *shift* sets of α are defined as

$$\begin{aligned} \operatorname{dom}(\alpha) &= \{x \in X_n : x\alpha = y \text{ for any } y \in X_n\}, \\ \operatorname{im}(\alpha) &= \{y \in X_n : x\alpha = y \text{ for any } x \in \operatorname{dom}(\alpha)\}, \\ \operatorname{fix}(\alpha) &= \{x \in \operatorname{dom}(\alpha) : x\alpha = x\} \text{ and} \\ \operatorname{shift}(\alpha) &= \{x \in \operatorname{dom}(\alpha) : x\alpha \neq x\} = \operatorname{dom}(\alpha) \setminus \operatorname{fix}(\alpha), \end{aligned}$$

respectively, in common usage in semigroup theory. Moreover, for any $\emptyset \neq Y \subseteq X_n$, let $Y\alpha = \{y\alpha : y \in Y\}$ be the *image set of* Y *under* α, and $\alpha_{|_Y}$ be the restriction map of α to Y. In 1966, as quoted in [10], for any $\emptyset \neq Y \subseteq X$, Magill, in [12], investigated the semigroup of all transformations on X which leave Y invariant, say

$$S(X,Y) = \{\alpha \in T_X : Y\alpha \subseteq Y\}.$$

Then many more studies have been done about this semigroup and new semigroups inspired by or defined similarly to this semigroup (see, for example, [1, 6, 9, 10, 13–16]). In one such study, presented in [16], Toker and Ayık considered the semigroup

$$T_{(n,m)} = \{\alpha \in T_n : X_m\alpha = X_m\}$$

of T_n for $1 \leq m \leq n-1$ and they showed that

$$\operatorname{rank}(T_{(n,m)}) = \begin{cases} 2 & \text{if } (n,m) = (2,1) \text{ or } (n,m) = (3,2) \\ 3 & \text{if } (n,m) = (3,1) \text{ or } 4 \leq n \text{ and } m = n-1 \\ 4 & \text{if } 4 \leq n \text{ and } 1 \leq m \leq n-2 \end{cases}.$$

Sommanee used the notation $PG_m(n)$ for $T_{(n,m)}$, and obtained the rank of $PG_m(n)$ by using a different technique in [14]. Recently, Konieczny presented some algebraic properties of the semigroup $T_{\mathbb{S}(Y)}(X) = \{\alpha \in T_X : \alpha_{|_Y} \in \mathbb{S}(Y)\}$ for any subset Y of X and any subsemigroup $\mathbb{S}(Y)$ of T_Y in [9]. Inspired by the studies summarized above, we defined and also obtained the ranks of certain semigroups of transformations whose restrictions are elements of a given semigroup in [4]. When we reviewed the studies on this subject, the special subsemigroups obtained with additional restrictions of these semigroups also arused our curiosity. Now, let us give the definitions of certain subsemigroups that will be the subject of this paper.

Let X be a nonempty set with cardinality n and let Y be a nonempty subset of X with cardinality $1 \leq m \leq n-1$. Without loss of generality, we can consider the sets X_n and X_m, rather than X and Y, respectively. Thus, for $1 \leq m \leq n-1$, let

$$\begin{aligned} IS^f_{(n,m)} &= \{\, \alpha \in I_n : \alpha_{|_{X_m}} \in S_m;\ \operatorname{dom}(\alpha) \setminus X_m \subseteq \operatorname{fix}(\alpha)\,\}, \\ II^f_{(n,m)} &= \{\, \alpha \in I_n : \alpha_{|_{X_m}} \in I_m;\ \operatorname{dom}(\alpha) \setminus X_m \subseteq \operatorname{fix}(\alpha)\,\}, \\ PS^f_{(n,m)} &= \{\, \alpha \in P_n : \alpha_{|_{X_m}} \in S_m;\ \operatorname{dom}(\alpha) \setminus X_m \subseteq \operatorname{fix}(\alpha)\,\}, \\ PT^f_{(n,m)} &= \{\, \alpha \in P_n : \alpha_{|_{X_m}} \in T_m;\ \operatorname{dom}(\alpha) \setminus X_m \subseteq \operatorname{fix}(\alpha)\,\}, \\ PI^f_{(n,m)} &= \{\, \alpha \in P_n : \alpha_{|_{X_m}} \in I_m;\ \operatorname{dom}(\alpha) \setminus X_m \subseteq \operatorname{fix}(\alpha)\,\}, \\ PP^f_{(n,m)} &= \{\, \alpha \in P_n : \alpha_{|_{X_m}} \in P_m;\ \operatorname{dom}(\alpha) \setminus X_m \subseteq \operatorname{fix}(\alpha)\,\}, \end{aligned}$$

and let

$$\mathbf{R} = \{\, IS^{f}_{(n,m)},\, II^{f}_{(n,m)},\, PS^{f}_{(n,m)},\, PT^{f}_{(n,m)},\, PI^{f}_{(n,m)},\, PP^{f}_{(n,m)} \}.$$

Clearly each element in $\mathbf{R}$ is a semigroup. We call these semigroups by semigroups of transformations with idempotent complement whose restrictions are elements of a given semigroup. Note that, since

$$TS^{f}_{(n,m)} = \{\, \alpha \in T_n : \alpha_{|X_m} \in S_m;\ \mathrm{dom}\,(\alpha) \setminus X_m \subseteq \mathrm{fix}\,(\alpha) \,\} \cong S_m \text{ and}$$

$$TT^{f}_{(n,m)} = \{\, \alpha \in T_n : \alpha_{|X_m} \in T_m;\ \mathrm{dom}\,(\alpha) \setminus X_m \subseteq \mathrm{fix}\,(\alpha) \,\} \cong T_m,$$

with the required isomorphism $\alpha \mapsto \alpha_{|X_m}$, we exclude the semigroups $TS^{f}_{(n,m)}$ and $TT^{f}_{(n,m)}$ for $1 \leq m \leq n-1$. Consequently, in this paper, we will focus on semigroups in $\mathbf{R}$ and we will find the rank of each of these semigroups.

2. Preliminaries

First, we state the following well-known lemma (this lemma also stated in [4]) which is easy to prove and useful throughout this paper.

Lemma 1 *Let T be a subsemigroup of a semigroup S, and let $S \setminus T$ be an ideal of S. If A is a finite generating set of S, then $T \cap A$ is a finite generating set of T. Consequently,* $\mathrm{rank}\,(S) > \mathrm{rank}\,(T)$. □

Let $\alpha \in S_n$ with $\mathrm{shift}\,(\alpha) = \{a_1, \ldots, a_r\}$ $(2 \leq r \in \mathbb{Z}^+)$, $a_i\alpha = a_{i+1}$ for each $1 \leq i \leq r-1$ and $a_r\alpha = a_1$. In this case, α is called a cycle of length r (a r-cycle) and denoted by $\alpha = (a_1 \ \cdots \ a_r)$. In general, for any $\emptyset \neq Y \subseteq X_n$, the identity permutation on Y is denoted by 1_Y. Note that the identity permutation 1_{X_n} in S_n is the unique 1-cycle in S_n and denoted also by (1).

Let $\alpha \in T_n$ with $\mathrm{shift}\,(\alpha) = \{a\}$ and $a\alpha = b$ for any $a, b \in X_n$. In this case, α is denoted by $\alpha = \|a\, b\|$.

Let $\alpha \in I_n$ with $\mathrm{dom}\,(\alpha) = X_n \setminus \{a_r\}$, $\mathrm{shift}\,(\alpha) = \{a_1, \ldots, a_{r-1}\}$ for $r \in \mathbb{Z}^+ \setminus \{1\}$ and $a_i\alpha = a_{i+1}$ for each $1 \leq i \leq r-1$. In this case, α is called a *chain* of length r (a r-chain) and denoted by $[a_1 \ \cdots \ a_r]$. In particular, if $\mathrm{dom}\,(\alpha) = \mathrm{fix}\,(\alpha) = X \setminus \{a_1\}$, then α is called a 1-chain and denoted by $[a_1]$.

With the notations given above, note the well known facts (see, for example, [5, 7, 11]) that

$$S_2 = \langle (1\ 2) \rangle,\ \ T_2 = \langle (1\ 2), \|1\ 2\| \rangle,\ \ I_2 = \langle (1\ 2), [1] \rangle \text{ and } P_2 = \langle (1\ 2), \|1\ 2\|, [1] \rangle,$$

and that for $n \geq 3$,

$$\begin{aligned} &S_n = \langle (1\ 2), (1\ 2\ \cdots\ n) \rangle, && T_n = \langle (1\ 2), (1\ 2\ \cdots\ n), \|1\ 2\| \rangle, \\ &I_n = \langle (1\ 2), (1\ 2\ \cdots\ n), [1] \rangle \quad \text{and} && P_n = \langle (1\ 2), (1\ 2\ \cdots\ n), \|1\ 2\|, [1] \rangle. \end{aligned}$$

Moreover,

$$\mathrm{rank}\,(S_n) = \begin{cases} 1, & n = 2 \\ 2, & n \geq 3 \end{cases}, \quad \mathrm{rank}\,(T_n) = \mathrm{rank}\,(I_n) = \begin{cases} 2, & n = 2 \\ 3, & n \geq 3 \end{cases} \quad \text{and}$$

$$\mathrm{rank}\,(P_n) = \begin{cases} 3, & n = 2 \\ 4, & n \geq 3 \end{cases}.$$

An element e of a semigroup S is called an idempotent if $e^2 = e$. The set of all idempotents in any subset U of S is denoted by $E(U)$. It is well known that $\alpha \in P_n$ is an idempotent if and only if $\alpha_{|_{\mathrm{im}\,(\alpha)}} = 1_{\mathrm{im}\,(\alpha)}$, or equivalently, $\mathrm{im}\,(\alpha) = \mathrm{fix}\,(\alpha)$. Notice that $E(I_n)$ is a subsemigroup of I_n. Let us denote the free semilattice on X_n by $\mathcal{SL}(X_n)$ which is the semigroup of the power set of X_n with usual intersection of sets. If we define the map $\Phi : E(I_n) \to \mathcal{SL}(X_n)$ by $\alpha\Phi = \mathrm{dom}\,(\alpha)$ for all $\alpha \in E(I_n)$, then we see that $E(I_n)$ and $\mathcal{SL}(X_n)$ are isomorphic. Therefore, $|E(I_n)| = 2^n$, and moreover, $E(I_n) = \langle\, 1_{X_n}, [1], [2], \ldots, [n]\,\rangle$. Notice that $\mathcal{SL}(X_n)$ is generated by $\{X_n, X_{n1}, X_{n2}, \ldots, X_{nn}\}$ where $X_{ni} = X_n \setminus \{i\}$ for every $1 \leq i \leq n$. For further information on semigroup theory and transformation semigroups, we recommend referring to, for example, [5, 7].

Now, we give some notations (similarly defined in [4]) which will be useful throughout this paper. Let S be one of the semigroups in $\mathbf{R}$. Then let

$$\begin{aligned}
\Gamma(S) &= \{\, \alpha \in S : \alpha_{|_{X_n \setminus X_m}} = 1_{X_n \setminus X_m} \,\}, \\
\Gamma_m(S) &= \{\, \alpha \in S : \alpha_{|_{X_m}} = 1_{X_m}, \text{ and } (X_n \setminus X_m)\alpha \subseteq X_n \setminus X_m \,\},
\end{aligned}$$

which are clearly subsemigroups of S. Moreover, for any $\alpha \in S$, $\alpha = \alpha_{_{(1)}}\alpha_{_{(2)}}$ where $\alpha_{_{(1)}}$ and $\alpha_{_{(2)}}$ are the maps defined by

$$\begin{aligned}
i\alpha_{_{(1)}} &= \begin{cases} i\alpha & 1 \leq i \leq m \\ i & m+1 \leq i \leq n \end{cases} \quad \text{and} \\
i\alpha_{_{(2)}} &= \begin{cases} i & 1 \leq i \leq m \\ i\alpha & m+1 \leq i \leq n \end{cases},
\end{aligned}$$

respectively. Finally, for any $\alpha \in P_m$, let α^+ be the map defined by

$$i\alpha^+ = \begin{cases} i\alpha & 1 \leq i \leq m \\ i & m+1 \leq i \leq n \end{cases},$$

and, for any $\emptyset \neq U \subseteq P_m$, let $U^+ = \{\, \alpha^+ : \alpha \in U \,\}$.

3. Ranks of $IS^f_{(n,m)}$ and $II^f_{(n,m)}$

It is easy to prove that

$$IS^f_{(n,m)} \cong S_m \times E(I_{X_n \setminus X_m}) \quad \text{and} \quad II^f_{(n,m)} \cong I_m \times E(I_{X_n \setminus X_m})$$

with the required isomorphism: $\alpha \mapsto (\alpha_{|_{X_m}}, \alpha_{|_{X_n \setminus X_m}})$. Then, since $|I_k| = \sum_{k=0}^{n} \binom{n}{k}^2 k!$ for every $k \in \mathbb{Z}^+$, and $I_{X_n \setminus X_m} \cong I_{n-m}$, we immediately have

$$|IS^f_{(n,m)}| = m!\, 2^{n-m} \quad \text{and} \quad |II^f_{(n,m)}| = \left(\sum_{k=0}^{m} \binom{m}{k}^2 k! \right) 2^{n-m}.$$

For any $\alpha \in E(IS^f_{(n,m)})$, since $\alpha_{|_{X_m}} = 1_{X_m}$ and $\alpha_{|_{X_n \setminus X_m}} \in E(I_{X_n \setminus X_m})$, it follows that $E(IS^f_{(n,m)}) \cong E(I_{X_n \setminus X_m})$. And since $E(I_{n-m}) \cong E(I_{X_n \setminus X_m})$ trivially, we have

$$|E(IS^f_{(n,m)})| = 2^{n-m}.$$

Similarly, we have

$$\begin{array}{rcl} \Gamma(IS^f_{(n,m)}) & \cong & S_m, \\ \Gamma(II^f_{(n,m)}) & \cong & I_m \text{ and} \\ \Gamma_m(IS^f_{(n,m)}) & = & \Gamma_m(II^f_{(n,m)}) \cong E(I_{X_n\setminus X_m}) \cong E(I_{n-m}) \end{array}$$

with the required isomorphisms defined as $\alpha \mapsto \alpha_{|X_m}$, $\alpha \mapsto \alpha_{|X_m}$ and $\alpha \mapsto \alpha_{|X_n\setminus X_m}$, respectively.

Lemma 2 *For $1 \le m \le n-1$, $II^f_{(n,m)} \setminus IS^f_{(n,m)}$ is an ideal of $II^f_{(n,m)}$.*

Proof For any $\alpha \in I_n$, it is clear that $\alpha \in II^f_{(n,m)} \setminus IS^f_{(n,m)}$ if and only if $\alpha_{|X_m} \in I_m \setminus S_m$. Furthermore, for any $\gamma \in II^f_{(n,m)}$, since $(\alpha\gamma)_{|X_m}, (\gamma\alpha)_{|X_m} \in I_m \setminus S_m$, we have $\alpha\gamma, \gamma\alpha \in II^f_{(n,m)} \setminus IS^f_{(n,m)}$, and so $II^f_{(n,m)} \setminus IS^f_{(n,m)}$ is an ideal of $II^f_{(n,m)}$. □

For any $1 \le r \le n$, recall that the 1-chain in I_n with domain $X_n \setminus \{r\}$ is denoted by $[r]$. Since we are concerned with both I_m and $I_{X_n\setminus X_m}$, to avoid confusion, we write $[1]_m$ if $[1] \in I_m$ and $[r]_{n,m}$ if $[r] \in I_{X_n\setminus X_m}$ for $1 \le m < r \le n$.

Theorem 1 *For $1 \le m \le n-1$, $IS^f_{(n,m)} = \langle \Gamma(IS^f_{(n,m)}) \cup \{[m+1], \ldots, [n]\}\rangle$, and moreover, we have*

$$\begin{array}{rcl} IS^f_{(n,1)} & = & \langle 1_{X_n}, [2], \ldots, [n]\rangle, \\ IS^f_{(n,2)} & = & \langle (1\,2), [3], \ldots, [n]\rangle, \\ IS^f_{(n,m)} & = & \langle (1\,2), (12\ldots m), [m+1], \ldots, [n]\rangle \textit{ for } m \ge 3. \end{array}$$

Proof Since $\Gamma(IS^f_{(n,m)}) \cong S_m$, it is enough to show that $IS^f_{(n,m)} = \langle \Gamma(IS^f_{(n,m)}) \cup \{[m+1], \ldots, [n]\}\rangle$. First, notice that, adapting from the well-known generating set of the $E(I_k)$ for any $k \in \mathbb{Z}^+$, we have immediately $E(I_{X_n\setminus X_m}) = \langle 1_{X_n\setminus X_m}, [m+1]_{n,m}, \ldots, [n]_{n,m}\rangle$. Then, since $\Gamma_m(IS^f_{(n,m)}) \cong E(I_{X_n\setminus X_m})$, we have

$$\Gamma_m(IS^f_{(n,m)}) = \langle 1_{X_n}, [m+1], \ldots, [n]\rangle.$$

For any $\alpha \in IS^f_{(n,m)}$, we have $\alpha = \alpha_{_{(1)}}\alpha_{_{(2)}}$, with $\alpha_{_{(1)}} \in \Gamma(IS^f_{(n,m)})$ and $\alpha_{_{(2)}} \in \Gamma_m(IS^f_{(n,m)})$. Therefore, since $1_{X_n} \in \Gamma(IS^f_{(n,m)})$, $\alpha \in \langle \Gamma(IS^f_{(n,m)}) \cup \{[m+1], \ldots, [n]\}\rangle$, and so $IS^f_{(n,m)} = \langle \Gamma(IS^f_{(n,m)}) \cup \{[m+1], \ldots, [n]\}\rangle$, as required. □

Corollary 1 *For $1 \le m \le n-1$, we have*

$$\operatorname{rank}(IS^f_{(n,m)}) = \begin{cases} n-m+1 & \textit{for } m = 1,2 \\ n-m+2 & \textit{for } m \ge 3 \end{cases}.$$

Proof Let U be an arbitrary generating set of $IS^f_{(n,m)}$. For any $1 \le i \le n-m$, since $[m+i] \in IS^f_{(n,m)} = \langle U\rangle$, there exist $\sigma_1, \ldots, \sigma_k \in U$ $(k \in \mathbb{Z}^+)$ such that $[m+i] = \sigma_1 \cdots \sigma_k$. Moreover, since $|\operatorname{im}(\alpha\beta)| \le \min\{|\operatorname{im}(\alpha)|, |\operatorname{im}(\beta)|\}$ for all $\alpha, \beta \in P_n$, we have $\sigma_1, \ldots, \sigma_k \in S_n \cup K_{n-1}$ where

$$K_{n-1} = \{\alpha \in IS^f_{(n,m)} : |\operatorname{dom}(\alpha)| = |\operatorname{im}(\alpha)| = n-1\},$$

and there exists at least one $1 \leq j \leq k$ such that $\sigma_j \in K_{n-1}$. Without loss of generality, suppose that $|\mathrm{dom}\,(\sigma_l)| = n$ for each $1 \leq l \leq j-1$. Since $\mathrm{dom}\,(\alpha\beta) \subseteq \mathrm{dom}\,(\alpha)$ for all $\alpha, \beta \in P_n$, and since $[m+i], \sigma_j \in K_{n-1}$, it follows that $\mathrm{dom}\,(\sigma_j) = \mathrm{dom}\,([m+i])$. That is, for each $1 \leq i \leq n-m$, there exists $\sigma_{j(i)} \in U$ such that $\mathrm{dom}\,(\sigma_{j(i)}) = \mathrm{dom}\,([m+i]) = X_n \setminus \{m+i\}$. Moreover, since $|\mathrm{dom}\,(\sigma_{j(i)})| = n-1$ for each $1 \leq i \leq n-m$, we must have $U \cap S_n \neq \emptyset$. For $m = 1$ or $m = 2$ the result is clear from Theorem 1. For $m \geq 3$, since

$$IS^f_{(n,m)} \setminus \Gamma(IS^f_{(n,m)}) = \{\alpha \in IS^f_{(n,m)} : \mathrm{dom}\,(\alpha) \neq X_n\}$$

is an ideal of $IS^f_{(n,m)}$, and since $\Gamma(IS^f_{(n,m)}) \cong S_m$, it follows from Lemma 1 that $\mathrm{rank}\,(IS^f_{(n,m)}) \geq n-m+2$, and so the result follows from Theorem 1. □

Theorem 2 *For $1 \leq m \leq n-1$, we have $II^f_{(n,m)} = \langle IS^f_{(n,m)} \cup \{[1]\}\rangle$.*

Proof First, recall that $I_m = \langle S_m \cup \{[1]_m\}\rangle$. Then, since $\Gamma(II^f_{(n,m)}) \cong I_m$, similarly we have $\Gamma(II^f_{(n,m)}) = \langle S^+_m \cup \{[1]\}\rangle$, where S^+_m is defined as in the section Preliminaries. For any $\alpha \in II^f_{(n,m)}$, we have $\alpha = \alpha_{_{(1)}}\alpha_{_{(2)}}$, with $\alpha_{_{(1)}} \in \Gamma(II^f_{(n,m)})$ and $\alpha_{_{(2)}} \in IS^f_{(n,m)}$. Therefore, since $S^+_m \subseteq IS^f_{(n,m)}$, we have $\alpha \in \langle IS^f_{(n,m)} \cup \{[1]\}\rangle$, and so $II^f_{(n,m)} = \langle IS^f_{(n,m)} \cup \{[1]\}\rangle$, as required. □

Corollary 2 *For $1 \leq m \leq n-1$, we have*

$$\mathrm{rank}\,(II^f_{(n,m)}) = \begin{cases} n-m+2 & \textit{for } m = 1,2 \\ n-m+3 & \textit{for } m \geq 3 \end{cases}.$$

Proof The result follows from Lemma 1, Lemma 2, Corollary 1, and Theorem 2. □

4. Ranks of $PS^f_{(n,m)}$, $PT^f_{(n,m)}$, $PI^f_{(n,m)}$ and $PP^f_{(n,m)}$

First, notice that, for each $V_m \in \{S_m, T_m, I_m, P_m\}$,

$$\begin{aligned} \Gamma(PV^f_{(n,m)}) &\cong V_m, \\ \Gamma_m(PV^f_{(n,m)}) &\cong E(I_{X_n \setminus X_m}) \end{aligned}$$

with the required isomorphisms $\alpha \mapsto \alpha_{|_{X_m}}$ and $\alpha \mapsto \alpha_{|_{X_n \setminus X_m}}$, respectively. Now, we state a lemma which can be proved easily with the isomorphism defined by $\alpha \mapsto (\alpha_{|_{X_m}}, \alpha_{|_{X_n \setminus X_m}})$.

Lemma 3 *For $1 \leq m \leq n-1$,*

$$\begin{aligned} &PS^f_{(n,m)} = IS^f_{(n,m)} \cong S_m \times E(I_{n-m}), \\ &PT^f_{(n,1)} = PS^f_{(n,1)} = IS^f_{(n,1)} \cong S_1 \times E(I_{n-1}) \cong E(I_{n-1}), \\ &PT^f_{(n,m)} \cong T_m \times E(I_{n-m}) \quad \textit{for } m \geq 2, \\ &PI^f_{(n,m)} = II^f_{(n,m)} \cong I_m \times E(I_{n-m}), \\ &PP^f_{(n,1)} = PI^f_{(n,1)} = II^f_{(n,1)} \cong I_1 \times E(I_{n-1}) \textit{ and} \\ &PP^f_{(n,m)} \cong P_m \times E(I_{n-m}) \quad \textit{for } m \geq 2. \end{aligned}$$

□

For each $1 \leq m \leq n-1$, since the ranks of $IS^f_{(n,m)}$ and $II^f_{(n,m)}$ are given in Corollaries 1 and 2, respectively, we consider only the subsemigroups $PT^f_{(n,m)}$ and $PP^f_{(n,m)}$ for $2 \leq m \leq n-1$.

Theorem 3 *For $2 \leq m \leq n-1$, we have*

$$PT^f_{(n,m)} = \langle\, PS^f_{(n,m)} \cup \{\, \|1\ 2\|\, \}\,\rangle \quad \textit{and}$$
$$PP^f_{(n,m)} = \langle\, PT^f_{(n,m)} \cup \{\, [1]\, \}\,\rangle = \langle\, PS^f_{(n,m)} \cup \{\, \|1\ 2\|,\ [1]\, \}\,\rangle.$$

Proof First, recall that $T_m = \langle S_m \cup \{\|1\ 2\|_m\}\rangle$ and $P_m = \langle T_m \cup \{[1]_m\}\rangle$. Since $\Gamma(PT^f_{(n,m)}) \cong T_m$ and $\Gamma(PP^f_{(n,m)}) \cong P_m$, similarly, we have

$$\Gamma(PT^f_{(n,m)}) = \langle S^+_m \cup \{\, \|1\ 2\|\, \}\rangle \quad \text{and} \quad \Gamma(PP^f_{(n,m)}) = \langle T^+_m \cup \{\, [1]\, \}\rangle.$$

For any $\alpha \in PT^f_{(n,m)}$, we have $\alpha = \alpha_{(1)}\alpha_{(2)}$, with $\alpha_{(1)} \in \Gamma(PT^f_{(n,m)})$ and $\alpha_{(2)} \in PS^f_{(n,m)}$. Since $S^+_m \subseteq PS^f_{(n,m)}$, it follows that $\alpha \in \langle PS^f_{(n,m)} \cup \{\, \|1\ 2\|\, \}\rangle$, and so $PT^f_{(n,m)} = \langle PS^f_{(n,m)} \cup \{\, \|1\ 2\|\, \}\rangle$.

For any $\alpha \in PP^f_{(n,m)}$, we have $\alpha = \alpha_{(1)}\alpha_{(2)}$, with $\alpha_{(1)} \in \Gamma(PP^f_{(n,m)})$ and $\alpha_{(2)} \in PT^f_{(n,m)}$. Since $T^+_m \subseteq PT^f_{(n,m)}$, it follows that $\alpha \in \langle PT^f_{(n,m)} \cup \{\, [1]\, \}\rangle$, and so $PP^f_{(n,m)} = \langle PT^f_{(n,m)} \cup \{\, [1]\, \}\rangle$. □

Lemma 4 *For $1 \leq m \leq n-1$,*

(i) $PT^f_{(n,m)} \setminus PS^f_{(n,m)}$ *is an ideal of* $PT^f_{(n,m)}$ *and*

(ii) $PP^f_{(n,m)} \setminus PS^f_{(n,m)}$ *is an ideal of* $PP^f_{(n,m)}$.

Proof It is a routine matter to prove as in Lemma 2. □

Corollary 3 *For $2 \leq m \leq n-1$, we have*

$$\operatorname{rank}(PT^f_{(n,m)}) = \begin{cases} n-m+2 & \textit{for } m = 2 \\ n-m+3 & \textit{for } m \geq 3 \end{cases}.$$

Proof The result follows from Lemma 1, Lemma 4 (i), the fact $PS^f_{(n,m)} = IS^f_{(n,m)}$, Corollary 1 and Theorem 3. □

Recall that, for any $\alpha \in P_n$, $\ker(\alpha) = \{(x,y) \in X_n \times X_n : x, y \in \operatorname{dom}(\alpha) \text{ and } x\alpha = y\alpha \text{ or } x, y \notin \operatorname{dom}(\alpha)\}$ is an equivalence relation on X_n and the equivalence classes of $\ker(\alpha)$ are all different preimage sets of elements in $\operatorname{im}(\alpha)$ together with $X_n \setminus \operatorname{dom}(\alpha)$ which form a partition of X_n. Also, recall that, $\ker(\alpha) \subseteq \ker(\alpha\beta)$ for any $\alpha, \beta \in P_n$. It is easy to see that, for $\alpha \in T_n$, if $|\operatorname{im}(\alpha)| = n-1$, then there exist $i \neq j \in X_n$ such that $\ker(\alpha)$ is the equivalence relation on X_n generated by $\{(i,j)\}$ which is denoted by $\ker(\alpha) = \{(i,j)\}^e$.

Theorem 4 *For $2 \leq m \leq n-1$, if U is a generating set of $PP^f_{(n,m)}$, then*

(i) *U must contain an arbitrary generating set of* $PS^f_{(n,m)}$,

(ii) *there exists an element* $\alpha \in U \cap (PP^f_{(n,m)} \setminus PT^f_{(n,m)})$ *such that* $|\mathrm{dom}\,(\alpha)| = |\mathrm{im}\,(\alpha)| = n-1$,

(iii) *there exist an element* $\beta \in U \cap (PT^f_{(n,m)} \setminus PS^f_{(n,m)})$ *and two elements* $1 \le i \ne j \le m$ *such that* $\ker(\beta) = \{i,j\}^e$.

Proof For $1 \le m \le n-1$, let $\emptyset \ne U \subseteq PP^f_{(n,m)}$ be a generating set of $PP^f_{(n,m)}$.

(i) From Lemmas 1 and 4 (ii), we immediately conclude that U must contain an arbitrary generating set of $PS^f_{(n,m)}$.

(ii) Consider the map $[i] \in PP^f_{(n,m)}$ for any $1 \le i \le m$. Since U is a generating set of $PP^f_{(n,m)}$, then there exist $\sigma_1, \ldots, \sigma_k \in U$ $(k \in \mathbb{Z}^+)$ such that

$$[i] = \sigma_1 \cdots \sigma_k.$$

Since $|\mathrm{dom}\,([i])| = n-1$ and since $\mathrm{dom}\,(\psi\theta) = (\mathrm{im}\,(\psi) \cap \mathrm{dom}\,(\theta))\psi^{-1}$ for any $\psi, \theta \in P_n$, there exists at least one $1 \le t \le k$ such that $|\mathrm{dom}\,(\sigma_t)| = n-1$. Notice also that, $|\mathrm{im}\,(\sigma_j)| \ge n-1$ for each $1 \le j \le k$ since $|\mathrm{im}\,([i])| = n-1$ and $\mathrm{im}\,([i]) = \mathrm{im}\,(\sigma_1 \cdots \sigma_k)$, and so $|\mathrm{im}\,(\sigma_t)| = n-1$. Therefore, there exists at least one element $\alpha \in U \cap (PP^f_{(n,m)} \setminus PT^f_{(n,m)})$ such that $|\mathrm{dom}\,(\alpha)| = |\mathrm{im}\,(\alpha)| = n-1$.

(iii) Consider the map $\psi \in PT^f_{(n,m)} \setminus PS^f_{(n,m)}$ with $\ker(\psi) = \{i_0, j_0\}^e$ for any two elements $1 \le i_0 \ne j_0 \le m$. Then there exist $\omega_1, \ldots, \omega_l \in U$ $(l \in \mathbb{Z}^+)$ such that

$$\psi = \omega_1 \cdots \omega_l$$

and $n-1 \le |\mathrm{im}\,(\omega_r)| \le n$ for each $1 \le r \le l$. Since $\ker(\omega) \subseteq \ker(\omega\theta)$ for any $\omega, \theta \in P_n$, $\ker(\omega_1) \subseteq \ker(\omega_1 \cdots \omega_l) = \ker(\psi) = \{i_0, j_0\}^e$ and $n-1 \le |\mathrm{im}\,(\omega_r)| \le n$ for each $1 \le r \le l$, we conclude that there exists $1 \le s \le l$ and $1 \le i_s \ne j_s \le n$ such that $\omega_s \in U \cap (PT^f_{(n,m)} \setminus PS^f_{(n,m)})$ and that $\ker(\omega_s) = \{i_s, j_s\}^e$. Without loss of generality we can suppose that ω_s is the first element in the sequence $\omega_1, \ldots, \omega_l$ satisfying this condition. In this case also notice that $\omega_1, \ldots, \omega_{s-1}$ must be in $PS^f_{(n,m)}$, $i_0(\omega_1 \cdots \omega_{s-1}) \ne j_0(\omega_1 \cdots \omega_{s-1})$, $i_s \ne j_s$, and that

$$\{i(\omega_1 \cdots \omega_{s-1}), j(\omega_1 \cdots \omega_{s-1})\} = \{i_s, j_s\}.$$

Since $X_m\psi \subseteq X_m$ then we have $i_s, j_s \in X_m$, and so there exist an element $\beta \in U \cap (PT^f_{(n,m)} \setminus PS^f_{(n,m)})$ and two elements $1 \le i \ne j \le m$ such that $\ker(\beta) = \{i,j\}^e$. $\square$

Corollary 4 *For* $2 \le m \le n-1$, *we have*

$$\mathrm{rank}\,(PP^f_{(n,m)}) = \begin{cases} n-m+3 & \text{for } m = 2 \\ n-m+4 & \text{for } m \ge 3 \end{cases}.$$

Proof The result follows from the fact that $PS^f_{(n,m)} = IS^f_{(n,m)}$, Corollary 1 and Theorems 3 and 4. $\square$

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